The Scalar Tensor Fourth Order Gravity: solutions, astrophysical applications and conformal transformations

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Abstract

The first step of comparison between the General Relativity and the so-called Scalar Tensor Fourth Order Gravity must be performed in the same framework of astrophysical applications that demonstrated the validity of Einsteins theory with respect to the Newtonian mechanics.

By considering other curvature invariants and also the presence of non-minimally coupled scalar field the solutions for the spherically symmetric systems are shown in the limit of weak field. The solutions are applied in the astrophysical framework: the galactic rotation curves and the gravitational lensing. Finally the mathematical relations between the Jordan and Einstein frames in the weak field limit are shown.
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References

Scalar Tensor Fourth Order Gravity (STFOG)

The action

\[ A = \int d^4x \sqrt{-g} \left[ f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi;_{\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_{\text{matter}} \right] \]

- \( R \rightarrow \) Ricci scalar
- \( R_{\alpha\beta}R^{\alpha\beta} \equiv Y \rightarrow \) Ricci tensor square
- \( \phi \rightarrow \) scalar field
- \( \mathcal{L}_{\text{matter}} \rightarrow \) minimally coupled ordinary matter
- \( g \rightarrow \) determinant of metric tensor \( g_{\mu\nu} \)
- \( \mathcal{X} = \frac{8\pi G}{c^2} \) but we use \( c = 1 \)
- \( R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}, R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \ldots \rightarrow \text{convention} \)
- \( 2\Gamma^\mu_{\alpha\beta} = g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) \)
- \((+---) \rightarrow \) the adopted signature
The field equations by applying $\delta(\cdot) \rightarrow \delta g_{\mu\nu} \frac{\delta(\cdot)}{\delta g_{\mu\nu}} + \delta \phi \frac{\delta(\cdot)}{\delta \phi}$

\[
\begin{align*}
f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi;\alpha \phi;\alpha}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu} \Box f_R + 2 f_Y R_{\mu}^{\alpha} R_{\alpha \nu} - 2 [f_Y R_{(\mu};\nu)_{\alpha}] \\
+ \Box [f_Y R_{\mu\nu}] + [f_Y R_{\alpha \beta}]^{\alpha \beta} g_{\mu\nu} + \omega(\phi)\phi;\mu \phi;\nu = \chi T_{\mu\nu}
\end{align*}
\]

\[
2 \omega(\phi) \Box \phi + \omega(\phi)\phi;\alpha \phi;\alpha - f_\phi = 0
\]

\[
f_R R + 2 f_Y R_{\alpha \beta} R^{\alpha \beta} - 2 f + \Box [3 f_R + f_Y R] + 2 [f_Y R^{\alpha \beta}]_{\alpha \beta} - \omega(\phi)\phi;\alpha \phi;\alpha = \chi T
\]

- $T_{\mu\nu} = - \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}$ → the tensor of matter
- $T = T^\rho_{\rho}$ → trace of matter tensor
- $f_R = \frac{df}{dR}$, $f_Y = \frac{df}{dY}$, $\omega(\phi) = \frac{d\omega}{d\phi}$, $f_\phi = \frac{df}{d\phi}$, $\Box = :\sigma ;\sigma$

The scalar field $\phi$ is coupled with the geometry $R$, $R_{\mu\nu}$ (as the classical scalar tensor gravity) and the field equations are differential equations of fourth order.
The technically possible frameworks of STFOG

The first two interesting frameworks are

<table>
<thead>
<tr>
<th>Newtonian and Post-Newtonian limit</th>
<th>Weak field limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{\mu\nu} \sim \left(1 + g_{tt}^{(2)}(t, x) + g_{tt}^{(4)}(t, x) \right.$</td>
<td>$g_{\mu\nu} \sim \eta_{\mu\nu} + h_{\mu\nu}(x)$</td>
</tr>
<tr>
<td>$g_{ti}^{(3)}(t, x) \right.$</td>
<td>$\phi \sim \phi^{(0)} + \phi^{(2)}(t, x)$</td>
</tr>
<tr>
<td>$- \delta_{ij} + g_{ij}^{(2)}(t, x)$</td>
<td>$\mu \sim , \mu + \tilde{\gamma}<em>{\mu} = -\nabla + \tilde{\gamma}</em>{\mu}$</td>
</tr>
<tr>
<td>$\square \sim -\Delta + \tilde{\square}$</td>
<td>$\square \sim \square_{\eta}$</td>
</tr>
</tbody>
</table>

Slow motion in the spherically symmetric systems

- $x = (t, x)$
- The approximation level is driven by an expansion of the powers of $1/c$
- The approximation level is driven by the linearization of field equations $O(h_{\mu\nu})^2 \ll 1$
- In the Newtonian limit the space time is time independent

Gravitational waves
The Newtonian limit of STFOG

By setting $f_R(0, 0, \phi(0)) = 1$, $\omega(\phi(0)) = 1/2$ the field equations are

$$\left(\triangle - m_Y^2\right)\triangle\Phi + \left[\frac{m_Y^2}{2} - \frac{m_R^2 + 2m_Y^2}{6m_R^2}\triangle\right]R + m_Y^2 f_{R\phi}(0, 0, \phi(0)) \triangle\phi = -m_Y^2 \chi T_{tt}$$

$$\left(\triangle - m_R^2\right)R - 3m_R^2 f_{R\phi}(0, 0, \phi(0)) \triangle\phi = m_R^2 \chi T$$

$$\left(\triangle - m_\phi^2\right)\phi + f_{R\phi}(0, 0, \phi(0)) R = 0$$

where

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + 2\Phi(x) & 0 \\ 0 & -\delta_{ij} \end{pmatrix}$$

The derivatives are calculated on the Minkowskian background

$$m_R^2 \equiv -\frac{1}{3f_{RR}(0, 0, \phi(0)) + 2f_Y(0, 0, \phi(0))}$$

$$m_Y^2 \equiv \frac{1}{f_Y(0, 0, \phi(0))}$$

$$m_\phi^2 \equiv -f_{\phi\phi}(0, 0, \phi(0))$$

- $T_{tt} = T = \rho(x)$ → the energy mass density of the source
- $T = T^\rho_\rho$ → trace of matter tensor
- The theory is parameterized by the coefficients of Taylor expansion of Lagrangian.
The scalar field $\varphi$ and also the Ricci scalar $R$ are auxiliary fields. The physical outcome is

$$\Phi(x) = -\frac{GM}{|x|} \left[ 1 + g(\xi, \eta) e^{-mR \tilde{k}R |x|} + \left[ 1/3 - g(\xi, \eta) \right] e^{-mR \tilde{k}R |x|} - \frac{4}{3} e^{-mY |x|} \right]$$

where $g(\xi, \eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}$, $\tilde{k}_R^2 = \frac{1}{2} \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}$, $\eta = \frac{m_\phi}{m_R}$ and $\xi = 3f_{R\phi}(0, 0, \phi(0))^2$.

The effective Lagrangian of STFOG is given as

$$R + \frac{f_{RR}(0, 0, \phi(0))}{2} R^2 + \frac{f_{\phi}(0, 0, \phi(0))}{2} \varphi^2 + f_{R\phi}(0, 0, \phi(0)) R \phi + f_Y(0, 0, \phi(0)) R_{\alpha \beta} R^{\alpha \beta} + \frac{\left| \nabla \phi \right|^2}{2}$$

... and if we add the terms proportional to $R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta}$ and/or to $\Box^k R$? **The outcome is the same!!**

- The Gauss-Bonnet invariant ...
- $\Box^k$ is the four divergence ...
## The Newtonian limit of STFOG: Classes of the gravitational potentials (A, B)

<table>
<thead>
<tr>
<th>Case</th>
<th>Gravitational potential</th>
<th>Free parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[- \frac{GM}{</td>
<td>x</td>
</tr>
<tr>
<td>B</td>
<td>[- \frac{GM}{</td>
<td>x</td>
</tr>
</tbody>
</table>

- case A → $f(R)$
- case B → $f(R, R_{\alpha\beta} R^{\alpha\beta})$
- case C → $f(R, \phi) + \omega(\phi)\phi;\alpha \phi;^\alpha$
- case D → $f(R, R_{\alpha\beta} R^{\alpha\beta}, \phi) + \omega(\phi)\phi;\alpha \phi;^\alpha$
### The Newtonian limit of STFOG: Classes of the gravitational potentials (C)

<table>
<thead>
<tr>
<th>Gravitational potential</th>
<th>Free parameters</th>
</tr>
</thead>
</table>
| \[-\frac{GM}{|x|} \left[ 1 + g(\xi, \eta) e^{-m_R \tilde{k}_R |x|} + 
  + \frac{1}{3} - g(\xi, \eta) \right] e^{-m_R \tilde{k}_\phi |x|} \right] \] | \[m_R^2 = -\frac{1}{3f_{RR}(0,1)} \] |
| \[\xi = \frac{3f_{R\phi}(0,1)^2}{2\omega(1)} \] | \[m_\phi^2 = -\frac{f_{\phi\phi}(0,1)}{2\omega(1)} \] |
| \[\eta = \frac{m_\phi}{m_R} \] | \[g(\xi, \eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}} \] |
| \[\tilde{k}_{R,\phi}^2 = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2} \] |
The Newtonian limit of STFOG: Classes of the gravitational potentials (D)

<table>
<thead>
<tr>
<th>Gravitational potential</th>
<th>Free parameters</th>
</tr>
</thead>
</table>
| \(- \frac{GM}{|x|} \left[ 1 + g(\xi, \eta) e^{-m_R \tilde{k}_R |x|} + 
+ [1/3 - g(\xi, \eta)] e^{-m_R \tilde{k}_\phi |x|} - \frac{4}{3} e^{-m_Y |x|} \right] \) | \( m_R^2 = \frac{1}{3f_{RR}(0,0,1)+2f_Y(0,0,1)} \) |
| \( m_Y^2 = \frac{1}{f_Y(0,0,1)} \) |
| \( m_\phi^2 = -\frac{f_{\phi\phi}(0,0,1)}{2\omega(1)} \) |
| \( \xi = \frac{3f_{R\phi}(0,0,1)^2}{2\omega(1)} \) |
| \( \eta = \frac{m_\phi}{m_R} \) |
| \( g(\xi, \eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}} \) |
| \( \tilde{k}_{R,\phi}^2 = \frac{1-\xi+\eta^2 \pm \sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2} \) |
The Newtonian limit of STFOG: The crucial choice of the coefficient and of the theory!

The potential does not depend only on the masses propagating (cases A, B) but also on the function modeling the Yukawa corrections (case C, D).

Plot of coefficient \( g(\xi, \eta) \) (case C) with respect to quantity \( \xi \) for \( 0 \leq \eta \leq 0.99 \) with step 0.33.

\[
g(\xi, \eta) = \frac{1 - \eta^2 + \xi + \sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}{6\sqrt{\eta^4 + (\xi - 1)^2 - 2\eta^2(\xi + 1)}}, \quad k_{R,\phi}^2 = \frac{1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2}}{2}, \quad \eta = \frac{m_\phi}{m_R} \text{ and } \xi = \frac{3f_R\phi(0,0,\phi(0))}{2\omega(\phi(0))}.
\]
The rotation curve in STFOG

- **In STFOG the Gauss theorem is not verified!**
- The case of source ball-like is different from the pointlike one: the potential depends on the dimension of the source.
- Generally for any term $\propto \frac{m r}{r}$ we have a geometric factor multiplying the Yukawa term.
- In the case of a ball with radius $\Xi$ we find $F(m \Xi) = 3 \frac{m \Xi \cosh m \Xi - \sinh m \Xi}{m^3 \Xi^3}$

\[ \Phi_{\text{ball}}(x) = - \frac{GM}{|x|} \left[ 1 + g(\xi, \eta) F(m_R \tilde{k}_R \Xi) e^{-m_R \tilde{k}_R |x|} + \left[ 1/3 - g(\xi, \eta) \right] F(m_R \tilde{k}_\phi \Xi) e^{-m_R \tilde{k}_\phi |x|} - \frac{4}{3} F(m_Y \Xi) e^{-m_Y |x|} \right] \]

- The rotation curve of the body around the ball source is easily obtained $v_c(r) = \sqrt{r \frac{\partial \Phi_{\text{ball}}(r)}{\partial r}}$ where $|x| = r$.
- A famous and satisfactory potential for the galactic rotation curves is the Sanders potential $\Phi_{\text{SP}}(r) = - \frac{GM}{r} \left[ 1 + \alpha e^{-m_S r} \right]$ where $\alpha$ and $m_S$ are free parameters.
The rotation curve in STFOG and in General Relativity

Plot circular velocity by assuming for a ball source with mass $M$ and radius $\Xi$. The behaviours are the following: red line (case D), blue line (case C), yellow line (case B), green line (case A), magenta line (case of GR). The black lines are the Sanders model for $-0.95 < \alpha < -0.92$. The values of free parameters are: $\xi = -5, \eta = 0.3, m_Y = 1.5 \cdot m_R, m_S = 1.5 \cdot m_R, m_R = 0.1 \cdot \Xi^{-1}$
The rotation curve in FOG: The galactic rotation

- The model of the galaxy: bulge, disk and (why not) dark matter component
- The gauss theorem is out but the field equations are linear → Superposition principle

\[
\rho_{\text{bulge}}(r) = \frac{M_b}{2\pi \xi_b^3 - \gamma} \frac{e^{-r^2/\xi_b^2}}{r^2} e^{-r^2/\xi_b^2}
\]

\[
\sigma_{\text{disk}}(R) = \frac{M_d}{2\pi \xi_d^2} e^{-R/\xi_d}
\]

\[
\rho_{\text{DM}}(r) = \frac{\alpha M_{DM}}{\pi (4 - \pi) \xi_{DM}^3} \frac{1}{1 + \frac{r^2}{\xi_{DM}^2}}
\]

The galactic potential is performed by numerical integration

\[
\Phi(x) = \Phi(r, R, z) = -G \int d^3x' \frac{\rho(x')}{|x - x'|} \left[ 1 + \frac{1}{3} e^{-m_R|x - x'|} - \frac{4}{3} e^{-m_Y|x - x'|} \right]
\]

The rotation curve in the galactic plane → \( v_c(R) = \sqrt{R \frac{\partial}{\partial R} \Phi(R, R, 0)} \)
The rotation curve induced by

- bulge component: GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line).
- disk component: GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line)
The rotation curve in FOG: The galactic rotation curve

The rotation curve induced by

- DM component: GR + DM (dashed and dotted line), FOG + DM (dotted line).
- Galactic structure: GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line): GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line) and FOG + DM (dotted line)

\[ v_c(R) = \frac{GM(R)}{R} \]

\[ v_c(R) = \frac{GM(R)}{R} \]

\[ R \text{(Kpc)} \]

\[ v_c \text{(Km/s)} \]
The experimental galactic rotation curves vs FOG

Superposition of theoretical behaviors (GR (dashed line), GR + DM (dashed and dotted line), FOG (solid line), FOG + DM (dotted line)) on the experimental data for Milky Way and NGC 3198

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>$M_b$</th>
<th>$\xi_b$</th>
<th>$\gamma$</th>
<th>$M_d$</th>
<th>$\xi_d$</th>
<th>$M_{DM}$</th>
<th>$\xi_{DM}$</th>
<th>$\alpha$</th>
<th>$\Xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milky Way</td>
<td>0.77</td>
<td>0.5</td>
<td>1.5</td>
<td>5.20</td>
<td>3.5</td>
<td>1.68</td>
<td>5.5</td>
<td>0.50</td>
<td>20</td>
</tr>
<tr>
<td>NGC 3198</td>
<td>0</td>
<td>/</td>
<td>/</td>
<td>2.60</td>
<td>3.5</td>
<td>0.84</td>
<td>5.5</td>
<td>0.53</td>
<td>20</td>
</tr>
</tbody>
</table>

The units → $(10^{10} M_\odot, 1 \text{Kpc})$. The data fitting → $m_R = 10^{-2} \text{Kpc}^{-1}$, $m_\gamma = 10^2 \text{Kpc}^{-1}$

**We need Dark Matter component**
The gravitational lensing by extended matter distribution in FOG

The deflection angle of photon, by solving the geodesic motion, is given as

$$\vec{\alpha} = \int_{z_i}^{z_f} \left[ \nabla_\xi (\Phi + \Psi) + \hat{z} \partial_z (\Phi - \Psi) \right] dz$$

where $g^{(2)}_{ij} = \Psi \delta_{ij}$. In presence of the generic matter distribution we find

$$\begin{align*}
\Phi + \Psi &= -2G \int d^2 \xi' dz' \frac{\rho(\xi'',z')}{\Delta(\xi',\xi'',z,z')} + 2G \int d^2 \xi' dz' \frac{\rho(\xi'',z')}{\Delta(\xi',\xi'',z,z')} e^{-\mu \gamma \Delta(\xi',\xi'',z,z')} \\
\Phi - \Psi &= -\frac{2G}{3} \int d^2 \xi' dz' \frac{\rho(\xi'',z')}{\Delta(\xi',\xi'',z,z')} \left[ e^{-m_R \Delta(\xi',\xi'',z,z')} - e^{-m_Y \Delta(\xi',\xi'',z,z')} \right]
\end{align*}$$
The gravitational lensing by extended 2D matter distribution in FOG

Then the deflection angle

\[
\bar{\alpha} = -2G \int_{z_i}^{z_f} d^2 \xi' dz' \frac{\rho(\xi', z')(\xi - \xi')}{\Delta(\xi, \xi', z, z')} - 2G \int_{z_i}^{z_f} d^2 \xi' dz' \frac{\rho(\xi', z'')[1 + mY\Delta(\xi, \xi', z, z')]}{\Delta(\xi, \xi', z, z')} e^{-mY\Delta(\xi, \xi', z, z')} (\xi - \xi') \\
+ \frac{2G}{3} \int_{z_i}^{z_f} d^2 \xi' dz' \frac{\rho(\xi', z')(z - z'')}{\Delta(\xi, \xi', z, z')} \left[ \left( 1 + mR\Delta(\xi, \xi', z, z') \right) e^{-mR\Delta(\xi, \xi', z, z')} \right] \\
- \left( 1 + mY\Delta(\xi, \xi', z, z') \right) e^{-mY\Delta(\xi, \xi', z, z')} 
\]

If the lens belongs to the plane orthogonal to the direction of the photon (hypothesis of the "thin lens")

\[ f(R)\text{-gravity} = \text{General Relativity} \]

\[
\bar{\alpha} = 4G \int d^2 \xi' \Sigma(\xi') \left[ \frac{1}{|\xi' - \xi'|} - |\xi - \xi'| \mathcal{F}_{mY}(\xi, \xi') \right] \frac{\xi - \xi'}{|\xi - \xi'|} 
\]
The gravitational lensing by pointlike source in FOG

In the case of pointlike source \( i.e. \Sigma(\xi') = M \delta^{(2)}(\xi') \)

\[
\bar{\alpha} = 2 r_g \left[ \frac{1}{|\xi|} - |\xi| \mathcal{F}_{mY}(\xi, 0) \right] \frac{\xi'}{|\xi|}
\]

and the correction to the images position is given by \( \theta = \theta^{GR} \pm \frac{\theta_E^2}{\sqrt{\beta^2 + 4 \theta_E^2}} \mathcal{F}(\theta^{GR}) \theta^{GR} \) where

\[
\mathcal{F}(\theta) = \int_0^\infty dz \frac{(1 + mY D_{OL} \sqrt{\theta^2 + z^2})}{\sqrt{(\theta^2 + z^2)^3}} e^{-mY D_{OL} \sqrt{\theta^2 + z^2}}
\]

and \( \theta_E = \sqrt{\frac{2 r_g D_{LS}}{D_{OL} D_{OS}}} \) is the Einstein angle.
The Newtonian limit of the Scalar Tensor Gravity in the Jordan frame

The action of the scalar-tensor theory of gravity in the Jordan frame

\[ A^{\text{JF}} = \int d^4x \sqrt{-g} \left[ \phi R + V(\phi) + \omega(\phi) \phi;\alpha \phi;\alpha + \mathcal{X} \mathcal{L}_m \right] \]

The field equations for the fields \( g_{\mu\nu} \) and \( \phi \)

\[
\begin{align*}
\phi R_{\mu\nu} - \frac{\phi R + V(\phi) + \omega(\phi) \phi;\alpha \phi;\alpha}{2} g_{\mu\nu} + \omega(\phi) \phi;\mu \phi;\nu - \phi;\mu\nu + g_{\mu\nu} \Box \phi &= \mathcal{X} T_{\mu\nu} \\
2 \omega(\phi) \Box \phi + \omega(\phi) \phi;\alpha \phi;\alpha - RV_\phi(\phi) &= 0 \\
\phi R + 2V(\phi) + \omega(\phi) \phi;\alpha \phi;\alpha - 3 \Box \phi &= -\mathcal{X} T
\end{align*}
\]

The solutions in the case of pointlike source are

\[
\begin{align*}
\Phi(x) &= \frac{GM}{\phi(0)|x|} \left\{ 1 - \frac{e^{-m_\phi |x|}}{2 \omega(\phi(0)) \phi(0) - 3} \right\} \\
\psi(x) &= -\frac{GM}{\phi(0)|x|} \left\{ 1 + \frac{e^{-m_\phi |x|}}{2 \omega(\phi(0)) \phi(0) - 3} \right\} \\
\phi(x) &= \phi(0) - \frac{1}{2 \omega(\phi(0)) \phi(0) - 3} g |x| e^{-m_\phi |x|} \\
m_\phi^2 \equiv -\frac{\phi(0) V_{\phi\phi}(\phi(0))}{2 \omega(\phi(0)) \phi(0) - 3}
\end{align*}
\]
The action of the scalar-tensor theory of gravity in the Einstein frame

\[ \mathcal{A}^{EF} = \int d^4x \sqrt{-\tilde{g}} \left[ \Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi};\alpha \tilde{\phi}^{\alpha} + \chi \tilde{L}_m \right] \]

The field equations for the fields \( \tilde{g}_{\mu \nu} \) and \( \tilde{\phi} \)

\[ \Xi \tilde{R}_{\mu \nu} - \frac{\Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi};\alpha \tilde{\phi}^{\alpha}}{2} \tilde{g}_{\mu \nu} + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_{;\mu} \tilde{\phi}_{;\nu} = \chi \tilde{T}_{\mu \nu} \]

\[ 2 \tilde{\omega}(\tilde{\phi}) \tilde{\Box} \tilde{\phi} + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}^{;\alpha} \tilde{\phi}_{;\alpha} - W_{\tilde{\phi}}(\tilde{\phi}) - \chi \frac{\delta \tilde{L}_m}{\delta \tilde{\phi}} = 0 \]

\[ \Xi \tilde{R} + 2W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi};\alpha \tilde{\phi}^{\alpha} = -\chi \tilde{T} \]

By imposing the conformal transformation \( \tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu} \)

\[ \tilde{\omega}(\tilde{\phi}) d\tilde{\phi}^2 = \frac{\Xi}{2} [2 \phi \omega(\phi) - 3] \left( \frac{d\phi}{\phi} \right)^2, \quad W(\tilde{\phi}) = \frac{\Xi^2}{\phi(\tilde{\phi})^2} V(\phi(\tilde{\phi})) , \]

\[ \tilde{L}_m = \frac{\Xi^2}{\phi(\tilde{\phi})^2} L_m \left( \frac{\Xi \tilde{g}_{\rho \sigma}}{\phi(\tilde{\phi})} \right), \quad \phi \Omega^{-2} = \Xi \]
The Newtonian limit of the Scalar Tensor Gravity in the Einstein frame

- If \( \tilde{\omega}(\tilde{\phi}) = -1/2, \Xi = 1 \) and \( \omega(\phi) = -\omega_0/\phi \) the links between the two frames in the newtonian limit are

\[
\bar{\phi}(\phi) = \phi_0 + \sqrt{2\omega_0 + 3} \ln \phi, \quad W(\bar{\phi}) = \exp \left( -\frac{2\bar{\phi}}{\sqrt{2\omega_0 + 3}} \right) V \left( e^{\frac{\tilde{\phi}}{\sqrt{2\omega_0 + 3}}} \right)
\]

\[
\tilde{L}_m = 2\rho \exp \left( -\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}} \right), \quad \phi \Omega^{-2} = 1
\]

- The solutions generated again by the pointlike source

\[
\tilde{\phi} = \frac{-GM}{|x|}, \quad \tilde{\psi} = \tilde{\phi}
\]

\[
\tilde{\phi} = \sqrt{2\omega_0 + 3} \ln \phi^{(0)} + \frac{1}{\phi^{(0)} \sqrt{2\omega_0 + 3}} \frac{r_g}{|x|} e^{-m_\phi |x|}
\]

- The difference between the gravitational potentials in the Jordan and Einstein frame is obtained

\[
\bar{\phi} - \phi^{(0)} \phi = \frac{\phi}{2} = \frac{GM}{2\omega_0 + 3} \frac{e^{-m_\phi |x|}}{|x|}
\]
Conclusions

- Any $f(R)$-Gravity contributes to the more attractive force, but the Ricci tensor invariant contributes to the anti-Gravity: we have a lower theoretical rotation velocity (with respect to the pure $f(R)$-Gravity), while the experimental evidence says opposite.

- From the point of view of gravitational lensing we have a perfect agreement between $f(R)$-Gravity and General Relativity in the case of thin lens. Only by adding $f(R_{\rho\sigma}R^{\rho\sigma})$ in the action we induce the modifications, but we do not find the hoped behavior.

- In the galactic dynamics we studied the motion of massive particles and in this case we find the corrections induced also by only $f(R)$-Gravity.

- The galactic rotation curves need a Dark Matter component, but the choice of framework is crucial for the space model of Dark Matter.

- Moreover if we consider $f(R, R_{\rho\sigma}R^{\rho\sigma})$-Gravity for the gravitational lensing we need a bigger amount of Dark Matter then in General Relativity.

- The behavior of anti-Gravity is similar to the Sanders potential.

- The weak conformal transformations: the correspondence of the solutions in the Newtonian limit between the Jordan and Einstein frame

- The effective Lagrangian in the Newtonian Limit: Scalar Tensor Fourth Order Gravity (STFOG)

- Finally we conclude that Dark matter effect as a pure single geometric phenomenon remains a hard challenge.

Thanks for your attention