THE GEOMETRY OF QUANTIZATION (II)

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MOTIVATIONS

- **Superposition** of states is one of the main feature of Quantum Mechanics (QM). It is at the heart of typical quantum mechanics effects like interference, entanglement etc.

- In **Classical Mechanics** (CM) instead we do not have superpositions and this creates all the troubles we encounter when we try to couple a quantum mechanical system with a classical one.

- In this work we shall try to understand what kills the superposition in Classical Mechanics. This will shed a new light not only on CM but also on QM.
Path integral counterpart of the operatorial formalism mentioned above (CPI), which is different from the quantum one (QPI);

Geometrical analysis of the space where the path integral formulation lives. Introduction of two Grassmann partners $\theta$ and $\bar{\theta}$ to time $t$ and of supervariables;

Superposition principle in KvN?

Local invariances;

Observables;

Solution of the problem.
Classical Mechanics

(Review of Hamiltonian formalism)

- Phase space: $\varphi^a \equiv (q_1, \ldots, q_n; p_1, \ldots, p_n), a = 1, \ldots, 2n$
- Hamiltonian: $H(\varphi)$
- Equations of motion: $\dot{\varphi}^a = \omega^a b_H$
- Poisson brackets: $\{A_1(\varphi), A_2(\varphi)\}_{pb} = \partial_a A_1 \omega^a b_A_2$
- Probability density distribution in phase space:
  - microcanonical $\rho(\varphi^a, t) \propto \delta[H - E]$
  - canonical $\rho(\varphi^a, t) \propto \exp[-\beta H]$
- Evolution of observables:
  \[
  \frac{dO}{dt} = \frac{\partial O}{\partial t} + \{O, H\}_{pb}
  \]
- Evolution of probability density:
  \[
  \frac{d\rho}{dt} = 0 \implies \frac{\partial \rho}{\partial t} = -\{\rho, H\}_{pb} = -\partial_b \rho \omega^a b_h H
  \]
Classical Mechanics in Operatorial Form

Koopman, von Neumann (1931)

- $\frac{\partial \rho}{\partial \phi} = -(\rho H) - i \omega^a \partial_a \rho = -i \hat{L} \rho$
  $\hat{L} = i \partial_a H \omega^a - i \partial_q H \partial_p - i \partial_p H \partial_q$ (Liouvilian)

- $\rho(\phi^a, \Delta t) \sim \rho(\phi^a - \omega^a \partial_a H \Delta t; 0)$
  Same functional form but different arguments.

- Propagator:
  $\rho(\phi, t) = \int d\phi_i K(\phi; \phi_i, t) \rho(\phi_i, t_i) \delta(\phi_f - \phi_{cl}(t_f; \phi_i, t_i))$
  $\phi_{cl}(t; \phi_i, t_i)$ is a solution of Hamilton equations of motion.
1) KvN introduced a Hilbert space made up of complex and square integrable functions on the phase space \( \psi(\varphi) \in L^2 \) and imposed the following two postulates:

2): \([\hat{q}, \hat{p}] = 0\)

3): \(|\psi(\varphi)|^2 = \rho(\varphi)\)

   Probability density in the phase space.

4): The evolution of \( \psi \) is via the Liouvillian \( \hat{L} \)

   \[
   \frac{\partial}{\partial t} \psi = \hat{L} \psi
   \]

   \[\hat{L} = i(\partial_t \mathcal{H}) \hat{\omega} \partial_t \hat{\omega} = \hat{\omega} \partial_t \hat{\omega} - \hat{\omega} \partial_{\hat{t}} \hat{\omega}\]

   As \( \hat{L} \) is linear in the derivatives one can derive:

   \[
   \frac{\partial \rho}{\partial t} = \hat{L} \rho
   \]

5): The observables?
Classical Dynamics

Same equation!

\[
\begin{align*}
  \frac{\partial}{\partial t} \psi(\varphi, t) &= \hat{L} \psi(\varphi, t) \\
  \frac{\partial}{\partial t} \rho(\varphi, t) &= \hat{L} \rho(\varphi, t)
\end{align*}
\]

Why? \( \hat{L} \) is linear in the derivatives.

Quantum Dynamics

Different Equations

\[
\begin{align*}
  \frac{\partial}{\partial t} \psi(x, t) &= \hat{H} \psi(x, t) \\
  \frac{\partial}{\partial t} \rho(x, t) &= -\frac{\hbar}{2m} \left( \hat{\nabla} \psi \hat{\nabla} \psi^* - \psi \hat{\nabla} \psi^* \right)
\end{align*}
\]

Reason: \( \hat{H} \) is second order in \( \partial_x \)
As the evolution of the $\psi$ and the $\rho$ is the same, the propagator will be the same:

$$K(\phi_f; t_f | \phi_i; t_i) = \delta[\phi_f - \phi_{cl}(t_f, \phi_i)]$$

$$K(\phi_f; t_f | \phi_i; t_i) = \lim_{N \to \infty} \left\{ \prod_{j=1}^{N-1} \int d\phi_j [\phi_j - \phi_{cl}(t_j, \phi_i)] \right\} \delta[\phi_f - \phi_{cl}(t_f, \phi_i)]$$

$$\delta[\phi_j - \phi_{cl}(t_j, \phi_i)] = \delta[\phi^a - \omega^{ab} \partial_b H]_{t_j}$$

$$\det[\delta^a_0 - \partial_0 (\omega^{ab} \partial_b H)]_{t_j}$$

$$\int d\lambda_a e^{i \lambda_a [\dot{\phi}^a - \omega^{ab} \partial_b H]} \int d\bar{c}_a dc_a e^{-i \bar{c}_a [\delta^a_0 - \partial_0 (\omega^{ab} \partial_b H)]c_a}$$

So in the continuum limit we get

$$K(\phi_f; t_f | \phi_i; t_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}c \mathcal{D}\bar{c} \exp i \int dt \bar{c}$$

$$\bar{c} = \lambda_a [\dot{\phi}^a - \omega^{ab} \partial_b H] + i \bar{c}_a [\delta^a_0 - \partial_0 (\omega^{ab} \partial_b H)]c_a$$
Path Integral and Operatorial Formalism

- \( \hat{L} = \lambda_\varphi \dot{\varphi}^2 + i \dot{\varphi} \varphi^4 - \mathcal{H} \), where:
  \[ \mathcal{H} = \lambda_\varphi \omega^2 \partial_\varphi \mathcal{H} + i \dot{\varphi} \omega^2 \partial_\varphi \mathcal{H} e^\varphi \]

  How can we turn \( \mathcal{H} \) into an operator?

- Commutators:
  \[ \langle [\varphi^a(t), \lambda_b(t)] \rangle = \lim_{\varepsilon \to 0} \langle \varphi^a(t + \varepsilon) \lambda_b(t) - \lambda_b(t + \varepsilon) \varphi^a(t) \rangle \]

  With our \( \hat{L} = \lambda_\varphi \dot{\varphi}^2 + i \dot{\varphi} \varphi^4 + \ldots \) the only non-zero commutators are:
  \[ \langle [\varphi^a(t), \lambda_b(t)] \rangle = i \delta^a_b; \quad \langle [\varphi^a(t), c_b(t)] \rangle = \delta^a_b \]

- A particular realization of the commutator above is:
  \[ \lambda_b \longrightarrow -i \partial \frac{\partial}{\partial \varphi^b}, \quad \bar{c}_b \longrightarrow \frac{\partial}{\partial \varphi^b} \]

  and \( \varphi^a, \bar{\varphi}^a \) as multiplicative operators.

- In particular \( \langle [\varphi^a(t), \varphi^b(t)] \rangle = 0 \) which confirms that we are doing classical mechanics.
Path Integrals --&gt; Operators

\[ \lambda_\alpha(t) \longrightarrow -\frac{\partial}{\partial \phi_i} \]

\[ \mathcal{H} = \lambda_\alpha \omega^a \partial_b H \longrightarrow \mathcal{H} = -\omega^a \partial_b H \partial_\alpha = \hat{L} \]

(Liouville operator)

\[ \int D\phi D\lambda \exp i \int dt \hat{L} \longrightarrow \exp -i\mathcal{H}t = \exp -iLt \]

Path integral

Operator formulation

(Koopman-von Neumann 1931)

**Question:** If the bosonic part of \( \hat{L} \) is related to the Liouville operator, what is the meaning of the full \( \hat{L} \)?

\[ \hat{L} = \lambda_\alpha [\omega^a - \omega^{ab} \partial_b H] + i\bar{c}_a [\delta^a_b \partial_b - \partial_b (\omega^{ac} \partial_c H)] c^b \]
\[ \bar{L} = \lambda_a [\dot{\varphi}^a - \omega^{ab} \partial_b H] + \bar{\varphi}_a [\delta_c^a \partial_b - \partial_b (\omega^{ac} \partial_c H)] \bar{c}^b \]

Equations of motion:

- Variation with respect to \( \lambda \) gives
  \[ \dot{\varphi}^a - \omega^{ab} \partial_b H = 0 \]
  standard equations of motion for \( \varphi \)

- Variation with respect to \( \bar{c} \) gives
  \[ \dot{c}^a - \omega^{ab} \partial_b H c^b = 0 \]

\( c^a \) have the same evolution equation as the first variations of \( \varphi^a \)

\[ (\delta \dot{\varphi}^a) - \omega^{ab} \partial_b H (\delta \varphi^b) = 0 \]

Jacobi fields.

\( \mathcal{H} \) generates the evolution of both the points of the phase space and their first variations.

\[ \langle \varphi(t) \varphi(0) \rangle \sim \exp \lambda t \]

Lyapunov exponents.
c and differential forms

- Infinitesimal time evolution:
  \[
  \begin{aligned}
  \varphi'^t &= \varphi + \epsilon \omega^a \partial_a tH \\
  \varphi'^t &= \varphi + \epsilon \omega^a \partial_a tH e^a = \frac{\partial \varphi^t}{\partial t^a}
  \end{aligned}
  \]

  $e^a$ transform as a basis for forms; $e^a \mapsto d\varphi^a \wedge d\varphi^b$  

  $e^a \in \text{basis of } T^*_a\mathcal{M}$

- $\mathcal{F}^{a_1} = \frac{1}{P} F_{a_1 \cdots a_n} d\varphi^{a_1} \wedge \cdots \wedge d\varphi^{a_n} \mapsto \mathcal{F} \equiv \frac{1}{P} F_{a_1 \cdots a_n} e^{a_1} \cdots e^{a_n}$

  forms on $\mathcal{M}$ \quad functions of $\varphi$ and $c$

- $\partial_1 = \partial_a - \epsilon \omega^{ab} \partial_b H = \frac{\partial \varphi}{\partial x^a}$

  $\partial_1$ transform as a basis for vector fields

- $\mathcal{V}^{a_1} = \frac{1}{P} V^{a_1 \cdots a_n} \partial_{a_1} \wedge \cdots \wedge \partial_{a_n} \mapsto \mathcal{V} \equiv \frac{1}{P} V^{a_1 \cdots a_n} \partial_1 \cdots \partial_1$

  antisymmetric tensors \quad functions of $\varphi$ and $\partial_1$
Global symmetries

Universal Symmetries (conserved charges for any system)

\[ Q_\text{RHS} \equiv i e^a \lambda; \quad \bar{Q}_\text{RHS} \equiv i \bar{e} \omega^{ab} \lambda; \quad Q_i \equiv e^i \bar{e}_a \]

\[ \bar{K} = \frac{1}{2} \omega_{ab} \bar{e}^b; \quad K = \frac{1}{2} \omega^{ab} e_a \]

Dynamical Symmetries (they depend on \( \bar{Q} \))

\[
\begin{align*}
N_H & = e^a \partial_a H(\phi) \\
\bar{N}_H & = \bar{e} \omega^{ab} \partial_b H(\phi) \\
Q_{\phi} & = Q_{\text{RHS}} - N_H \\
\bar{Q}_{\phi} & = Q_{\text{RHS}} + N_H
\end{align*}
\]

algebra N=2 supy:

\[ [Q_H, \bar{Q}_H] = 2 \imath \mathcal{H} \]

Square roots of the Hamiltonian:

\[
\begin{align*}
Q^{(i)} & = Q_{\text{RHS}} - N_H, \quad Q^{(i)} = Q_{\text{RHS}} + N_H \]

\[ Q^{(i)}_\phi = Q^{(i)} - \imath \mathcal{H} \]

If we apply \textbf{twice} the transformations generated by \( Q^{(i)} \), we get a time translation.
<table>
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<th>Differential Geometry Operations</th>
<th>Path Integral Correspondent</th>
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<tr>
<td>$dF^{(\sigma)}$</td>
<td>$[Q_{\alpha}, \hat{F}^{(\sigma)}]$</td>
</tr>
<tr>
<td>$\iota_v F^{(\sigma)}$</td>
<td>$[\hat{V}, \hat{F}^{(\sigma)}]$</td>
</tr>
<tr>
<td>$\mathcal{L}_h = d\iota_h + \iota_h d$</td>
<td>$\mathcal{H}$</td>
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<tr>
<td>$[d, \mathcal{L}_h] = 0$</td>
<td>$[Q_{\alpha}, \mathcal{H}] = 0$</td>
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<tr>
<td>$d_{eq} = d - \iota_h$</td>
<td>$Q_{(1)}$</td>
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- $dF^{(\sigma)}$: exterior derivative
- $\iota_v F^{(\sigma)}$: interior contraction
- $\mathcal{L}_h$: Lie derivative
- $d_{eq}$: equivariant exterior derivative
KvN did not specify which are the observables in their formulation.

In standard classical mechanics the observables are only:
\[ \hat{O}(\varphi) \]
So all the observables commutes and there is no interference effects.

If the observables were hermitian operators but functions also of \( \lambda, c, \bar{c} \)
\[ \hat{O}(\varphi, \lambda, c, \bar{c}) \]
then they would not commute and interference effects would appear because superposition is admitted in KvN.

The Way Out?.
Superspace and Superfields

Superspace:

\[ t \rightarrow (t, \theta, \bar{\theta}) \]

Base space:

\[ t \]

Target space:

\( \varphi, c, \bar{c}, \lambda \)

Superfield:

\[ \Phi^a(t, \theta, \bar{\theta}) = \phi^a(t) + \theta \epsilon^a(t) + \bar{\theta} \bar{\epsilon}^a(t) + i \theta \bar{\theta} \omega^{ab}_{\bar{c} \lambda}(t) \]
Comparison of Path Integrals

**Quantum** transition amplitude (QPI):

$$\langle q_f; t_f | q_i; t_i \rangle = \int D^\prime q Dp \exp \frac{i}{\hbar} \int dt L[q]$$

**Classical** transition amplitude (CPI):

$$\langle \Phi_f; t_f | \Phi_i; t_i \rangle = \int D^\prime \Phi D\phi \exp i \int d\tau d\bar{\tau} \bar{L}[\Phi]$$

The classical path integral has the same Lagrangian and the same measure as the quantum path integral but with the fields replaced by the superfields and the time integration replaced by the supertime integration.
Local universal symmetries

\[
\begin{align*}
\phi^a &\rightarrow \phi^a + \varepsilon(t) \theta^c \bar{c}^b \\
\phi^c &\rightarrow \phi^c - \varepsilon(t) \theta^a \\
\bar{c}^b &\rightarrow \bar{c}^b - \varepsilon(t) \bar{\theta} \bar{\omega}^{ab} \bar{\lambda}^a \\
\bar{\lambda}^a &\rightarrow \bar{\lambda}^a - \varepsilon(t) \omega^{ab} \phi^b
\end{align*}
\]

- The only quantity invariant under all three transformations above is \( \Phi^a \).
- So \( \int L(\Phi^a, \bar{\Phi}^a) \) is invariant and the transformations turn out to be local symmetries of our systems.
- As they are like gauge transformations, the only admissible observables are the following ones:
  \( \hat{O}(\Phi^a) \)
  which are gauge invariant.
• Let us go from the Heisenberg picture in $\theta, \bar{\theta}$ to the Schrödinger one
  
  $$e^{-\theta Q^{a\alpha}_{BS} - \bar{\theta} Q^{a\alpha}_{BS} \hat{O}_H(\Phi^a)} e^{\theta Q^{a\alpha}_{BS} + \bar{\theta} Q^{a\alpha}_{BS} \hat{O}_H(\Phi^a)} = \hat{\omega}^a(t)$$

• So $\hat{\omega}^a(t, \theta, \bar{\theta})$ is the Heisenberg picture, in $(\theta, \bar{\theta})$, of $\hat{\omega}^a(t)$

• Let us then turn the physical observables $\hat{O}(\Phi^a)$ into the Schrödinger picture form:
  
  $$e^{-\theta Q^{a\alpha}_{BS} - \bar{\theta} Q^{a\alpha}_{BS} \hat{O}_H(\Phi^a)} e^{\theta Q^{a\alpha}_{BS} + \bar{\theta} Q^{a\alpha}_{BS} \hat{O}_H(\Phi^a)} \equiv \hat{O}_S = O(\omega^a)$$

• We get in this way the standard observables $\hat{O}(\omega^a)$ of CM
• We have an operator $\hat{\phi}^a$ which commutes with all the observables $\hat{O}(\hat{\phi}^a)$ and which is not a multiple of the identity.

• This triggers a **superselection mechanism** which says that the Hilbert space of the system is given by the eigenvariety of the superselection operator $\hat{\phi}$:

  $\hat{\phi}^a |\phi^a_0\rangle = |\phi^a_0\rangle$

• Another eigenvariety will be:

  $\hat{\phi}^a |\phi^a_1\rangle = |\phi^a_1\rangle$
• As $|\phi_a^0\rangle$ and $|\phi_a^1\rangle$ belong to different Hilbert spaces we cannot superimpose them:
  
  $|\tilde{\phi}_a^a\rangle \equiv |\phi_a^0\rangle + |\phi_a^1\rangle$

  $|\tilde{\phi}_a^a\rangle$ is not a physical state.

• Not all hermitian operators are observables. The \textbf{non-observables} ones are all those hermitian operators which connect different superselected sectors. Example: the Liouvillian operator:

  $\hat{L} = \lambda_{\omega} \omega_{\rho} \partial_\rho H$

  which moves from one point to another.
Besides the observables we have to bring also the KvN-waves in the Schrödinger picture in $\theta, \bar{\theta}$

\[
\psi(t, \theta, \bar{\theta}) = e^{iQ_{\theta}^a, \bar{\theta} + \bar{Q}_{\theta}^a, \theta}\psi(t) = \psi(\phi^a + \theta \bar{\omega}^a_0, \bar{\phi})
\]

Note that $\tilde{c}^a$ commutes with all the observables so the system lives also in the eigenvariety:

\[
\tilde{c}^a |\phi^a_0\rangle = c^a_0 |\phi^a_0\rangle
\]

The physical states are then:

\[
\psi = \langle \phi^a|\tilde{\psi}\rangle|\phi^a_0\rangle = \delta(\phi - \phi_0)\delta(c - c_0)
\]
- Schroedinger picture

- Is \( \tilde{\psi} = \delta(\varphi - \varphi_0)\delta(c - c_0) \)
  of the form
  \[ \tilde{\psi}_S = \psi(\varphi^a + \theta c^a + \bar{\theta} \omega^{ab} \bar{c}_b, c^a) \] ?

- The only manner to achieve that is to put \( c = \bar{c} = 0 \) so:
  \[ \tilde{\psi}_S = \delta(\varphi - \varphi_0)\delta(c) \]
  and this is isomorphic to:
  \[ \psi_S = \delta(\varphi - \varphi_0) \]
  which are the physical states.
If we had neglected the differential forms in the KvN procedure, we would have had only functions of the \( \hat{\varphi}, \hat{\lambda} \). These would make up non-commuting observables \( O(\hat{\varphi}, \hat{\lambda}) \) and there would be no way to get rid of them via some principle but only via a postulate:

\[
\text{restrict the } O(\hat{\varphi}, \hat{\lambda}) \text{ to the } \tilde{O}(\hat{\varphi})
\]

The \( c, \bar{c} \) instead allow us to go to the superfield and have a gauge invariance which automatically selects the correct observables.

Role of \( c, \bar{c} \)
Quantization and superposition

- In this framework quantization is achieved by sending $\theta, \bar{\theta} \to 0$
  (identical to "geometric quantization").

- The "gauge invariance" disappears if $\theta, \bar{\theta} \to 0$

\[
\begin{align*}
\varphi^a &\to \varphi^a + \varepsilon(t) \theta c^a \\
\varphi^a &\to \varphi^a - \varepsilon(t) \bar{\theta} c^a
\end{align*}
\]

- Having no gauge invariance the superselection mechanism is not triggered anymore and we can have superposition.
We would like to get CM in the CPI formulation by starting from the path integral of QM and performing a “block spin” transformation to “block spin” of phase space of size $\gg \hbar$.

- We already passed from the QPI to the CPI via:
  \[ \phi^a \rightarrow \Phi^a \]
  so maybe the $\Phi^a$ represent the block spin:
  \[ \Phi^a(\varphi, c, \ldots) \]
  $c$ = side
  $\varphi$ = center of the block spin

- Points in phase space
- Block spin in phase space

QPI $\rightarrow$ CPI
If the new basic variables in CM are the blocks of phase space, then we can reparametrize its internal points as we like provided the block remain the same.
The local-transformation on the components of the superfields that we have used for the superselection rule are, "maybe", the above Internal Point Reparametrization.
We proved that the Lagrangians in the QPI and the CPI are the same, except for the exchange field-superfield, it means there is only a field “renormalization” and not a coupling and mass renormalization. This is typical of supersymmetric field theory.

Can we find an “analog” of the Gamma-function of the Callan-Symanzik renormalization group equation and its associated zeros?
$\gamma$ - function

CM?

Final Theory at Large distance?

QM?